Educ Stud Math
DOI 10.1007/s10649-017-9772-5

Does a transformation approach improve students' ability in constructing auxiliary lines for solving geometric problems? An intervention-based study with two Chinese classrooms

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Abstract We conducted an intervention-based study in secondary classrooms to explore whether the use of geometric transformations can help improve students' ability in constructing auxiliary lines to solve geometric proof problems, especially high-level cognitive problems. A pre- and post-test quasi-experimental design was employed. The participants were 130 eighth-grade students in two classes with a comparable background that were taught by the same teacher. A two-week intervention was implemented in the experimental class aiming to help students learn how to use geometric transformations to draw auxiliary lines in solving geometric problems. The data were collected from a teacher interview, video-recordings of the intervention, and pre- and post-tests. The results revealed that there was a positive impact of using geometric transformations on the experimental students' ability in solving high-level cognitive problems by adding auxiliary lines, though the impact on the students' ability in solving general geometric problems as measured using the overall average scores was not statistically significant. Recommendations for future research are provided at the end of the article.

Electronic supplementary material The online version of this article (doi:10.1007/s10649-017-9772-5) contains supplementary material, which is available to authorized users.

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Published online: 22 June 2017

Keywords Teaching of geometry · Geometric proof · Geometric transformation · Auxiliary lines · Intervention-based study

1 Background and rationale

Geometry has long occupied an important place in the mathematics curriculum in many countries. However, it has also proven to be a very difficult area for both teaching and learning, and has attracted increasing attention from mathematics education researchers, curriculum reformers and developers, as well as practitioners internationally over the recent decades. In China, the provision of geometry in the recent reformed curriculum has shifted the focus from the Euclidean axiomatic system to the use of three thematic approaches: "the nature of geometric figures", "transformation of geometric figures", and "geometric figures and coordinates" to present all the contents of geometry, through which it is expected that students will improve their learning of geometry (China Ministry of Education, 2012).

As widely recognized, proof is one of the most challenging parts in students' learning of geometry (Mariotti, 2006; McCrone & Martin, 2004; Senk, 1985; Weber, 2001). Different researchers have analyzed various possible reasons and studied possible ways to improve students' understanding of geometric proof (e.g., see Harel, 1999; Hodds, Alcock, & Inglis, 2015; Inglis & Alcock, 2012) and enhance their proof abilities (Golzy, 2008; Hoyles & Jones, 1998; Marrades, 2000), though overall, probably due to its difficulty, it is still largely under-researched, let alone resolved. Researchers are still looking for more solid solutions, especially from classroom-based studies, which is also evident in this special issue of *Educational Studies in Mathematics*.

In geometric proof, adding auxiliary lines is often helpful and in many cases necessary, especially in solving challenging or high-level cognitive geometric problems. In fact, the word "auxiliary" itself means offering additional help or support. However, to add auxiliary lines is also very difficult for many students (Herbst & Brach, 2006; Senk, 1985; You, 2009). As Chou, Gao, and Zhang (1994) argued, "adding auxiliary lines is one of the most difficult and tricky steps in the proofs of geometry theorems" (p. 2). In this study, we aimed to address students' difficulty in constructing auxiliary lines by introducing a transformation approach as an intervention in the geometric classroom, and hence to improve their ability in solving geometric problems and, in particular, in solving challenging or high-level cognitive geometric problems, as understandably students would have more difficulty in solving challenging or high-level cognitive problems.

Mathematically, it is well known that the concept of transformation has played a central role in modern geometry. In Klein's landmark writings about the Erlangen Program, geometry was viewed as "the study of the properties of a space that are invariant under a given group of transformation" (Bonotto, 2007; Jones, 2000). Klein's Erlangen Program provides a fundamental view and a new approach, not only in modern advanced geometry but also in school geometry, particularly in geometric proof. For example, as researchers have pointed out, any proof by congruence in the Euclidean tradition can be done by congruent transformations, such as rotation, translation and reflection, that preserve everything (e.g., as betweenness of points, midpoints, segments, angles and perpendicularity) between the pre-image and the image (Barbeau, 1988; Jones, 2000; Nissen, 2000; Ng & Tan, 1984; Willson, 1977).

In China, the contemporary school curriculum has introduced geometric transformation since 2001 when the government issued its new reformed national curriculum standards

(China Ministry of Education, 2001). The new curriculum put more emphasis on three types of congruent transformations: translation, rotation and reflection. Following the national mathematics curriculum, Chinese school textbooks have also introduced geometric transformation as a standard topic of geometry, and, in some cases, used it to help students understand geometric proof.

For example, one textbook series introduces students to the method which asks them to fold an isosceles triangle in half in order to prove the equality of two base angles in an isosceles triangle (Ma, 2014b). As shown in Fig. 1, through folding (reflection), students can easily find that the crease line AM divides the triangle into two congruent triangles. From this, they can realize that by adding an auxiliary line (the crease line AM), one can obtain congruent right-angled triangles, $\triangle ABM$ and $\triangle ACM$, and hence prove the equality of the two base angles. In this case, introducing the idea of geometric transformation, i.e., reflection, can help students better understand where and how to add an auxiliary line and hence solve the proof problem.

The introduction of geometric transformation in school mathematics is also reflected in the school curricula in other countries, for example, England and the US. In England, the national curriculum requires that secondary school students be taught in geometry to "identify properties of, and describe the results of, translations, rotations and reflections applied to given figures" at Key Stage 3 (Year 7–9) (Department for Education, 2013) and furthermore to "describe the changes and invariance achieved by combinations of geometrical transformation" at Key Stage 4 (Year 10–11) (Department for Education, 2014). In the US, a most influential curriculum document, NCTM's *Principles and Standards for School Mathematics*, emphasized that in the teaching of geometry, "instructional programs from prekindergarten through grade 12 should enable each and every student to … apply transformations and use symmetry to analyze mathematical situations" (National Council of Teachers of Mathematics , 2000, p. 41). The inclusion of transformation in the school curriculum is also endorsed in the US widely-adopted *Common Core State Standards for Mathematics*, which suggests that students should be taught to "verify experimentally the properties of rotations, reflections and translations" in Grade 8 (Common Core State Standards Initiative, 2009, p. 55).

Having briefly described the mathematics and curriculum background, we point out that, from the perspective of educational research, there are mainly two reasons for us to use a transformation approach to facilitate students in adding auxiliary lines in solving geometric problems.

The first reason concerns the practical value of the study. As aforementioned, the reformed national mathematics curriculum in China has placed more emphasis on the transformation

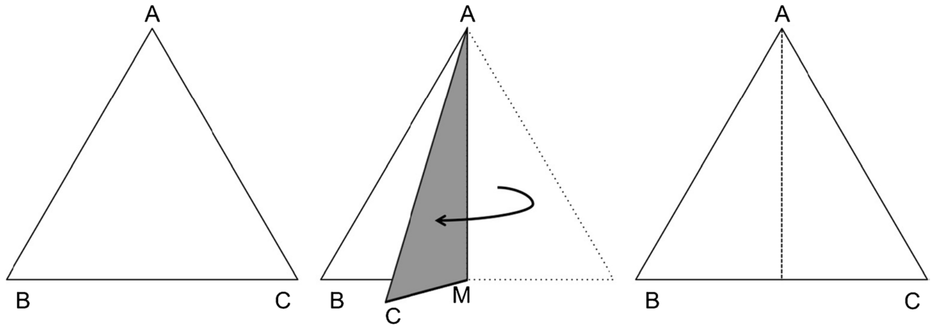


Fig. 1 Folding an isosceles triangle to prove its two base angles are equal

approach in presenting the contents of geometry to students. Because of the influence of the national mathematics curriculum on school textbooks and classroom teaching and learning, it is not only more feasible but also more meaningful and relevant than before from the curricular perspective for us to conduct the study exploring relevant issues concerning the use of a transformation approach in the teaching and learning of geometry.

It should also be noted that, along with the latest national curriculum development and reform, both researchers and school practitioners in China have shown increasing awareness about the possible positive impact of introducing geometric transformations on students' ability in constructing proofs. Accordingly, researchers have also discussed how transformations can be used to help geometry proofs, and, in particular, in adding auxiliary lines (Gao, 2010; Yang & Pan, 1996; Yao, 2010). For example, it has been argued that using a proper transformation can often rearrange a geometric diagram to form figures that students are familiar with (e.g., right-angled triangles, as shown in Fig. 1) while maintaining the properties concerned (e.g., the length of segments or the size of angles), thus helping students better understand where and how to add the auxiliary lines. Similarly, in a recent analysis about exercise problems in school textbooks, Wang (2010) argued that geometric transformations can provide ideas for students to look at all the known conditions and hence to add auxiliary lines in geometric proofs. However, there is a lack of research evidence from classroom-based studies, in China or elsewhere, about the effectiveness of using geometric transformation on students' ability in adding auxiliary lines, thus a reason for this study.

The second reason concerns the theoretical value of the study at a more general level beyond the Chinese educational setting. As we understand, since at least the early 1960s, researchers internationally have shown interest and offered various reasons for using a transformation approach in the teaching and learning of geometry. Researchers have argued that introducing a transformation approach can allow students to solve problems that might otherwise be more difficult (Hollebrands, 2003; Usiskin, 2014). In particular, some earlier studies conducted in the US have provided evidence suggesting that the use of transformation encouraged students to construct geometric proofs flexibly and creatively (Coxford, 1973; Usiskin, 1972; Usiskin & Coxford, 1972). However, we found that there have been virtually no studies, let alone intervention-based ones, on how transformation can be used to improve students' ability in constructing auxiliary lines (e.g., see Harel & Sowder, 2007). This fact is surprising as well as another motivating factor for us to conduct this study.

Given the relevant curricular and research background outlined above, we decided to conduct an intervention-based empirical study to explore the impact of introducing a transformation approach in the teaching and learning of geometry on students' ability in adding auxiliary lines for solving geometric problems. We were particularly interested to investigate the impact of the intervention on students' ability to solve challenging or high-level cognitive problems. The reason for this is twofold. First, solving high-level cognitive problems usually requires the use of more than one area of knowledge, including connections between different domains and an understanding of the terms of transformation (Brändström, 2005; Hiebert et al., 2003; Stein & Smith, 1998). Second, through transformation, students can move different parts of geometric figures together, find or visualize their relationships, and hence add auxiliary lines in solving geometric problems (Wang, 2010). In comparison, it appears clear that solving simple or low-level cognitive geometric problems is generally less problematic and adding auxiliary lines is usually not required, but, even when it is required, finding how to add auxiliary lines tends to be straightforward and accordingly does not need a transformation approach.

2 Research design and procedure

The study was carried out in the Chinese educational setting. A quasi-experimental design was used, consisting mainly of a classroom-based intervention, a pre-test and a post-test. In addition, interviews and video recordings were used in data collection.

2.1 Participants

The intervention took place in a western city in China. The selected institution was a junior secondary school covering grades 7, 8 and 9 (aged 12–15). There were ten classes for each of grades 7 and 9, and eight classes for grade 8, with 1560 students and 14 mathematics teachers in total. In many ways, it was a typical medium-size and average-performing secondary school in western China. It should be pointed out that the school had been actively involved in the reform of “Mathematics Classroom in Secondary Schools Based on the New Curriculum” and “Learning Plan Guidance”, which implies that they had some practical experiences about implementing the new curriculum, an important reason for our study to take place in the school.

Based on the school’s recommendation, two grade 8 classes, hereafter referred to as Class A and Class B, each with 65 students aged 13–14, were selected to participate in the study after necessary agreement was obtained from relevant parties. Both classes were taught by the same teacher. Grade 8 was selected because geometric proof was introduced in the school’s curriculum in this grade and it was more feasible to incorporate the classroom-based intervention into the curriculum. According to the participating teacher and the school, these two classes were comparable in terms of students’ academic background, as can also be seen from their average scores in two recent tests (see Table 1). Also, both of them were average classes compared to all grade 8 classes in the school. Class B was assigned as the control group and Class A was the experimental group. The students in the two classes received the same instruction except for Class A also receiving the intervention of geometric transformation.

The participating teacher, who as we noted was the mathematics teacher of both classes, has taught mathematics for 13 years using the textbook series published by Beijing Normal University Press (Ma, 2014b, a). She was an experienced teacher and had won awards including the title of “Teaching Master” at the provincial level. Before implementing the intervention, the researchers explained to the teacher the purpose and procedure of the study and the intervention. In particular, she was told to teach the two classes in the same way and as usual, except for introducing the intervention for the experimental class.

2.2 Pre-test and post-test

To compare students’ achievements before and after the intervention, both pre-test and post-test were designed and administered to the classes. The design of questions was based on the

Table 1 Average scores of two recent tests of the two classes

	Recent formative (monthly) test	Recent summative (mid-term) test
Class A (experimental; n = 65)	91.8	98.5
Class B (control; n = 65)	91.3	98.4

The total score for each test was 100 marks school curriculum (including the textbook used) as well as the consultation with the participating teacher, so they can be not only better integrated into the ongoing school curriculum but also better fit students' background of learning.

Table 2 shows the distribution of different types of questions in the pre-test (as well as in the post-test; see below). The total score of the pre-test was 40 marks, with the full marks for each question being 4 marks. A student could receive any mark, i.e., 0, 1, 2, 3 or 4, depending on their performance. Readers can access the full sets of the pre-test questions with scoring schemes in Online Resource 1.

As shown in Table 2, adding auxiliary lines was necessary for solving five questions in the pre-rest (PreQ1, PreQ2, PreQ8, PreQ9 and PreQ10), not necessary for solving two questions (PreQ6 and PreQ7), and optional for the remaining three questions (PreQ3, PreQ4 and PreQ5). In addition, four questions, i.e., PreQ3, PreQ5, PreQ7 and PreQ8, involved rotation: three (PreQ4, PreQ6 and PreQ10) were related to translation, one (PreQ9) was about reflection, and the remaining two (PreQ1 and PreQ2) did not involve any geometric transformation. The main reason for the test to include a small number of non-transformation questions and questions that do not require adding auxiliary lines to solve is for the test to be better integrated with the school curriculum and students' learning progress.

As also shown in Table 2, for comparison purposes, two types of questions designed in the pre-test were also provided in the post-test, including questions (PosQ8, PosQ9 and PosQ10) that involved geometric transformation and could not be solved without auxiliary lines, and questions (PosQ3 and PosQ4) that were not related to geometric transformation but could not be solved without auxiliary lines. In addition, questions PosQ6 and PosQ10 were related to rotation, PosQ5, PosQ8 and PosQ9 to reflection, and PosQ7 to translation. PosQ1 to PosQ 4 were not related to geometric transformation, and they were designed to explore the influence of the two-week intervention on students solving general geometry problems without the use of transformation and to better fit the school regular curriculum. In addition, the test included three challenging geometric questions, i.e., PosQ6, PosQ8 and PosQ10, which require high-level cognition to solve, as one of the study aims was to detect the influence of using geometric transformation on students' ability in adding auxiliary lines to solve high-level cognitive problems.

Table 2 Distribution of the questions in the pre-test and post-test

		PreQ1	PreQ2	PreQ3	PreQ4	PreQ5	PreQ6	PreQ7	PreQ8	PreQ9	PreQ10
Pre-test	Use of geometric transf.[a]	N	N	Y	O	Y	Y	Y	Y	Y	Y
		-	-	Ro	Tr	Ro	Tr	Ro	Ro	Re	Tr
	Adding aux. lines[b]	Y	Y	O	O	O	N	N	Y	Y	Y
		PosQ1	PosQ2	PosQ3	PosQ4	PosQ5	PosQ6	PosQ7	PosQ8	PosQ9	PosQ10
Post-test	Use of geometric transf.[a]	N	N	N	N	O	O	O	Y	Y	Y
		-	-	-	-	Re	Ro	Tr	Re	Re	Ro
	Adding aux. lines[b]	N	N	Y	Y	Y	Y	Y	Y	Y	Y

[a] Use of geometric transformation: *Y*– using geometric transformation is either necessary or highly expected as it would help solve the problem, *O*– using geometric transformation is optional, *N*– using geometric transformation is not necessary and not helpful, *Ro*– rotation, *Tr* translation, *Re*– reflection

[b] Adding auxiliary lines: *Y*– adding auxiliary lines is necessary, *O*– adding auxiliary lines is optional, *N*– adding auxiliary lines is not necessary and not helpful

The total score of the post-test was 56 marks, with 12 full marks for PosQ6, 8 full marks for each of PosQ7 and PosQ9, and 4 full marks for each of the remaining questions. Like in the pre-test, students could receive any mark from 0 to the full marks based on their solutions. The full sets of all the post-test questions with the scoring schemes can be found in Online Resource 2. In addition, examples of students' solutions, together with the marks given, can be also found in Online Resource 1 for the pre-test (e.g., PreQ2 and PreQ6) and Online Resource 2 for the post-test (e.g., PosQ5 and PosQ8).

2.3 Intervention

The instructional intervention used ten proof questions, which were designed specifically for this study. The participating teacher enacted those questions in the experimental group, i.e., Class A, and introduced how to use geometric transformation to find and construct auxiliary lines that could help solve these proof questions. The intervention was carried out in four lessons with each lasting 45 min, spreading over two consecutive weeks in June 2015.

In the ten intervention questions, five questions were related to rotation, four were related to reflection, and one was related to translation. It should be noted that, as in designing the pre- and post-test questions, to better incorporate the intervention into the existing teaching scheme in the participating class, the design and distribution of the intervention questions were also based on and hence constrained by the curriculum structure including textbooks used during the intervention, with more concentration on rotation and less on reflection and translation. This also explains why the three non-auxiliary lines-related questions were used. More specifically, given the students' knowledge based on the school curriculum they have learned, adding auxiliary lines is necessary for solving three of the intervention questions, one each on translation, rotation and reflection. It is optional for solving another four intervention questions, which were also related to rotation and reflection. For the remaining three questions, although adding auxiliary lines was not needed to solve them, the teacher also explained the thinking of geometric transformation in her instruction in the experimental class. Table 3 shows the information of the questions employed in the instructional intervention. Readers can also access all the intervention questions in Online Resource 3.

In lesson 1, the teacher used IntQ1 and IntQ2 to introduce the basic idea of geometric transformation and guide students to compare the figure before and after the transformation. In lesson 2, she used IntQ3, 4 and 5 to help students realize that the use of the three transformations could be helpful in constructing auxiliary lines and hence solving the geometric problems. In lesson 3, the teacher linked IntQ6, 7 and 8 to the previous five questions and

Table 3 Questions used in instructional intervention

	IntQ1	IntQ2	IntQ3	IntQ4	IntQ5	IntQ6	IntQ7	IntQ8	IntQ9	IntQ10
Use of geometric transformation[a]	O Ro	Y Re	O Ro	O Ro	O Re	Y Tr	Y Ro	Y Re	Y Ro	O Re
Adding auxiliary lines[b]	O	N	O	O	O	Y	Y	Y	N	N
Used in lessons	Lesson 1		Lesson 2			Lesson 3			Lesson 4	

[a] Use of geometric transformation: *Y*– using geometric transformation is either necessary or highly expected as it would help solve the problem, *O*– using geometric transformation is optional, *N*– using geometric transformation is not necessary and not helpful, *Ro*– rotation, *Tr*– translation, *Re*– reflection

[b] Adding auxiliary lines: Y− adding auxiliary lines is necessary; O− adding auxiliary lines is optional; N− adding auxiliary lines is not necessary and not helpful

led students to find the differences and similarities in the use of transformation and auxiliary lines. Finally, in lesson 4, she summarized the instructions of the whole intervention and emphasized the flexibility of geometric transformation in constructing auxiliary lines. Some classroom invention episodes as video-recorded are given in Section 3.2 below.

The four lessons took place during students' free periods in the afternoons with the consent of the school and the students. The learning environment was the same as in the control class in order to maintain the integrity of the study. The intervention instruction combined self-study, group discussion and teacher's demonstration, at about 20%, 40%, and 40%, respectively, in terms of class time. The first 10 min in every lesson was self-study time that was set to make students familiar with the selected questions. Then, students discussed the questions in groups. Finally, the teacher explained the solutions of those questions and explained to students the idea of geometric transformation. As mentioned earlier, all the intervention lessons were video-recorded and the teacher was also interviewed after the intervention was carried out.

2.4 Limitations

It should be stressed that the intervention was carried out in a relatively short duration, which presents both feasibilities and limitations. As researchers have argued, intervention of a short duration can help better control compounding variables and make incorporating the classroom-based intervention into existing curriculum structures more practical (Stylianides & Stylianides, 2013). In fact, when we communicated with the participating teacher and school, they agreed that a two-week duration was the "best fit" in terms of curriculum and school context (e.g., without affecting students' preparation for the forthcoming end-of-year examination).

On the other hand, we must remind the readers of the limitations the short duration also inevitably means that, in particular, the intervention only covered a limited scope of geometric topics from the school curriculum, and moreover it was not intended to detect the long-term effects on students' learning. In addition, in relation to the exploratory nature of the study, readers should also note that the sample of the study was from a particular school in China, and hence the results of the study should not be generalized to other students and school settings with different curricular, social and cultural backgrounds.

3 Findings and discussion

Quantitative methods including statistical analysis were chiefly used to analyze the numerical scores of students in the pre- and post-tests. Qualitative methods were used to analyze the test papers collected from the pre- and post-tests to obtain in-depth information about how students actually solved the geometric problems and to analyze the video data and interview transcripts.

3.1 Pre- and post-tests

The pre-test was carried out in the two selected classes, and received 64 answer sheets from Class A and 65 from Class B. Table 4 presents the average score of nine questions in both classes, sorted by the average score of Class A in descending order. The result shows that Class A did not appear to be much different from Class B. In particular, the two classes got the same scoring average in PreQ 3, 4 and 8, Because the data do not show normal distribution, the

Table 4 Students' average score on questions in pre-test

	PreQ1	PreQ5	PreQ3	PreQ4	PreQ10	PreQ6	PreQ2	PreQ8	PreQ9
Class A	4	4	3.92	3.92	3.68	3.64	3.12	2.96	2.80
Class B	3.96	3.8	3.92	3.92	3.32	3.60	2.80	2.96	2.64

Question 7 was excluded due to a technical error on the test paper, i.e., incorrect figure was given

Mann–Whitney *U* test was used to compare the difference between the two classes. The results show there was no significant difference at the 0.05 level (see Table 5; $p = 0.125$).

Further examining the pre-test questions, as mentioned earlier, adding an auxiliary line was optional to solve PreQ3 and PreQ4. However, all the students in both classes constructed their proofs based on the properties of parallelograms, and no one considered their proof from the perspective of geometric transformations, which we think was largely due to students' unfamiliarity with transformations and their use in solving such problems. It implies that students need to be explicitly taught in geometric transformation in order for them to use and apply such an approach in solving geometric problems.

Solving PreQ6 does not need any auxiliary line but it requires students to use the thinking of transformation to understand its reasoning process. For PreQ8, there were many potential relationships in its conclusion but most students only gave one possible answer, which resulted in a low score. Reflection was also an optional method to solve PreQ8, which was not found in students' answer sheets. The result again revealed that students rarely used transformation in geometric proof, which suggests that the instructional intervention provided in our study was meaningful.

Now let us turn to the post-test. As described earlier, the post-test consisted of 10 questions, and was administered in both classes after the intervention. In total, 58 answer sheets were received from Class A and 64 from Class B. Table 6 presents the students' average score, in terms of percentage from 0 (0%) to 1 (100%), on all the questions in descending order according to Class A. The percentage is used across all the questions for each comparison since different questions carried different full marks, ranging from 4 to 12, as mentioned earlier.

As it can be seen from Table 6, the experimental class outperformed the control class in 7 of the 10 questions in terms of the average scores. Nevertheless, the Mann–Whitney *U* test results also show that there was no significant difference between the two classes at the 0.05 level (see Table 7; $p = 0.357$).

As indicated earlier, we were particularly interested to know if the intervention would help students develop their ability in solving challenging or high-cognitive level geometric problems. For this purpose, we further classified all the questions in the post-test into two groups based on the cognitive levels required for solving the questions: one includes all the questions of ordinary cognitive level or general questions, and the other includes all the high-level cognitive questions or challenging questions. As a result, three questions in the post-test, i.e.,

Table 5 Mann–Whitney test results on pre-test scores

	n	Mean Rank	Rank Sum	*U*	*Z*	*P*
Class A	64	70.02	4481.28	1759	−1.534	0.125
Class B	65	60.06	3903.90			

Table 6 Students' average score on questions in post-test

	PosQ1	PosQ2	PosQ5	PosQ4	PosQ8	PosQ10	PosQ3	PosQ9	PosQ6	PosQ7
Class A	1	1	1	0.99	0.97	0.97	0.90	0.87	0.79	0.75
Class B	0.98	0.98	0.96	0.96	0.94	0.93	0.94	0.93	0.76	0.89

Each figure in this table is obtained by dividing the mean of the original scores of all the students in the class by the full marks of the question, for example, 0.98 (or 98%) for PosQ1 in Class B is obtained by 3.92 (the average score) ÷ 4 (the full marks). Note for some questions, the full marks were 8 or 12, while for the others the full marks were 4

PosQ6, 8 and 10, were classified as high-level cognitive questions. To solve these questions, students were required to use a mixture of different kinds of geometric knowledge, such as congruent triangles, the basic properties of quadrilaterals, the application of special angles and deduction of quantitative relationships. In addition, students were expected to identify relevant conditions in these questions and to link them to a transformation approach to effectively find the solutions.

Further analysis revealed that there existed considerable differences between the two classes in solving high-level cognitive questions in favor of the experimental class, suggesting that the intervention of introducing students to the transformation approach had a positive effect on students' ability in constructing auxiliary lines in solving these challenging problems. Next, we first take a look at PosQ6.

> PosQ6: In rhombus *ABCD*, $\angle ADC = 120^\circ$ and *E* is a point on the diagonal of *AC*; connecting point *D* with point *E*, there is $\angle DEC = 50^\circ$. Rotate *BC* 50 degrees around point *B* anticlockwise and extend it to intersect the extension of *ED* at point *G*.

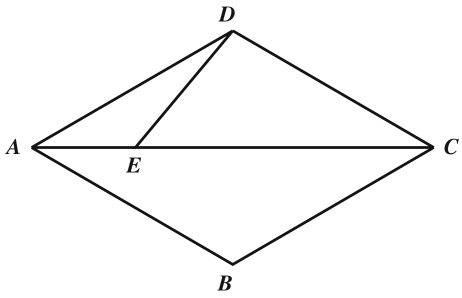
(1) Complete the figure (Fig. 2) according to the instruction above.
(2) Prove: *EG* = *BC*.
(3) Establish an equation to express the quantitative relationship between segments *AE*, *EG* and *BG*: ________________________.

This question was related to rotation. Students were required to do the construction according to their comprehension of the question (see Fig. 3). After analyzing the answer sheets, we found that there existed a remarkable difference between the experimental group (Class A) and the control group (Class B) in sub-question (3), a challenging one. It is a fill-in-the-blank question, which is usually more difficult than questions like multiple choice questions, as there is no hint in the question texts. In this case, compared with the control group, more students in the experimental group applied the transformation of rotation to solve the question correctly, e.g., rotating *AE* 50 degrees anticlockwise around E or rotating *DG* 30 degrees clockwise around G, and then proving the eq. $AE + BG = \sqrt{3EG}$. The average score

Table 7 Mann–Whitney test results on post-test scores

	n	Mean Rank	Rank Sum	*U*	*Z*	*P*
Class A	58	58.45	3390.10	2033	0.920	0.357
Class B	64	64.27	4113.28			

Fig. 2 Diagram for PosQ6



of Class A, in terms of percentage (with 1 being 100% of the full marks), was 0.64 while it was only 0.45 for Class B.

For PosQ8 shown below, the approach of geometric transformation was optional. However, using the approach (reflection) would help make the process of solving this question easier to students.

> PosQ8: The figure (see Fig. 4) shows quadrilateral *ABCD* with $AB = AD$, $\angle BAD = 120^{\circ}$, and $\angle B = \angle ADC = 90^{\circ}$. Point *E* is on line *BC* and point F is on line *CD*. $\angle EAF = 60^{\circ}$. Explain the relationship between segments *BE*, *EF* and *FD*.

Transformation is not explicitly mentioned in PosQ8, making it more difficult for students to think of using auxiliary lines. At the secondary level, there were two methods to solve this problem: one was to use the transformation, i.e., reflecting $\triangle ABE$ over the line *AE* to get $\triangle AEM$ (Fig. 5) or rotating $\triangle ABE$ anti-clockwise by 120^0 to get $\triangle ADG$ (Fig. 6); the other was to draw the height of $\triangle AEF$, i.e., *AM*. Both these ideas for solving this question were not so straightforward to students, but if students were aware of the approach of transformation (here reflection or rotation), it would help them identify where and how to add the auxiliary lines, and hence facilitate the solution. The question was designed and classified as a challenging question, as solving it required knowledge of congruent triangles (HL, SAS, SSS and AAS), transformation (the equality of corresponding angles and sides) and the properties of reflection or rotation. The percentage of students in Class A using rotation to solve the question was 96.6%, considerably higher than that in Class B at 89.1%.

A similar result in favor of the experimental class was found in PosQ10, as shown below.

Fig. 3 BG is obtained after the required rotation of BC

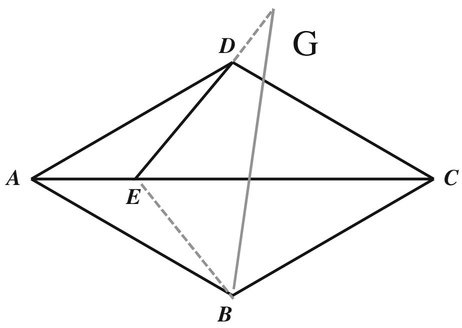
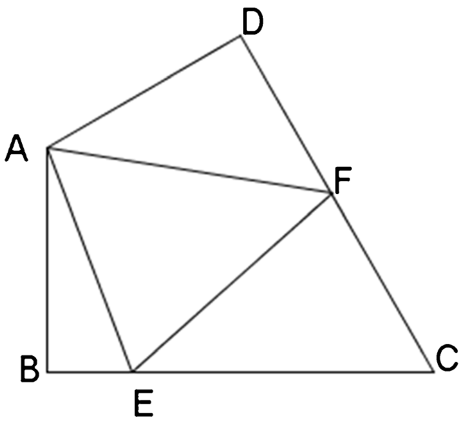


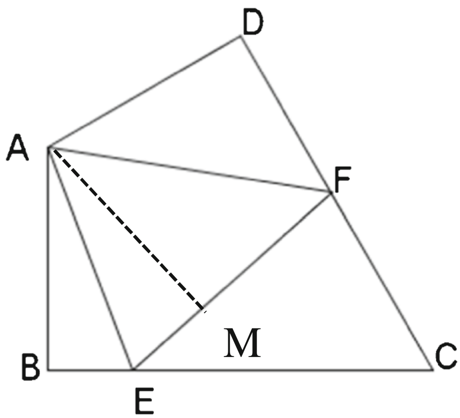
Fig. 4 Diagram for question 8



PosQ10: The figure (see Fig. 7) shows $a//b//c$. The distance between lines a and b is 3 while the distance between lines b and c is 1. The distance from point A to line a is 2 and the distance from point B to line c is 3; $AB = 2\sqrt{30}$. Please find a point M on line a and a point N on line c, so that $MN \perp a$ and the value of $AM + MN + NB$ is the minimum. In this case, the value of $AM + NB$ is _________.

This question was challenging as it did not offer any clue for students to use transformation, and they would need to reinterpret some information in terms of transformation; for example, "(the length) is the minimum" implies "the use of reflection". Once students constructed a parallelogram and applied the transformation of reflection to find point M and point N, they could easily solve the problem. Solving this question required students to use their knowledge about the properties of parallelograms, congruent triangles, and the facts that "the sum of any two sides of a triangle is greater than the third one" and "the shortest distance between two points is the length of the segment joining the two points". Accordingly, the most difficult part was to realize why and how auxiliary lines should be constructed to help solve the question, which required a high level of cognition or understanding about parallel lines and their properties. Although the average scores of the two classes were close in the post-test, an

Fig. 5 *AM* is the reflection image of AB over AE



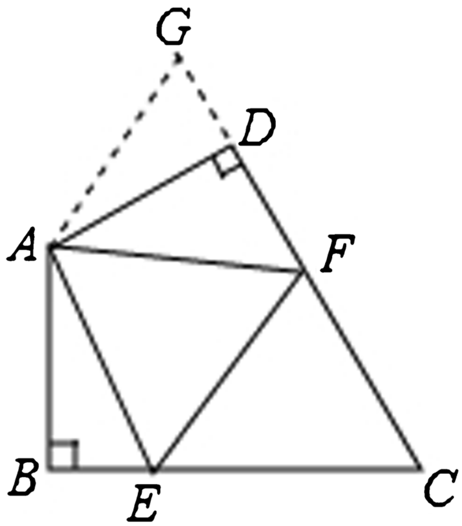


Fig. 6 △*ADG* is the rotational image of △*ABE*

important difference was detected in their ability to add auxiliary lines: 69% of the students in the experimental class added the auxiliary lines correctly, while the corresponding percentage in the control class was only 42%.

A further look at the data collected revealed that, although 13.8% of the students in the experimental class added auxiliary lines incorrectly, in the answers given most of them were trying to construct parallelograms. In contrast, in the control group, 40.1% of the students added auxiliary lines incorrectly and, moreover, most of them just added vertical lines and then connected the given points, and there was no clear sign of using translation or constructing a parallelogram. In conclusion, the students in the experimental class showed considerably better awareness of applying geometric transformation to add auxiliary lines, suggesting that this kind of awareness and ability should be better explicitly taught and developed through teachers' teaching in classroom.

3.2 Interview with the teacher

As described earlier, the post-intervention interview with the participating teacher, together with the video-records of the intervention sessions, was intended to gather data

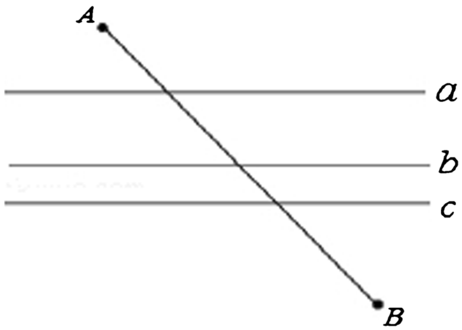


Fig. 7 Diagram for PosQ10

about the intervention itself and the teacher's view and observation about the result of intervention. The following conversation during the interview reveals how the intervention was implemented, which is also reflected in the video-records.

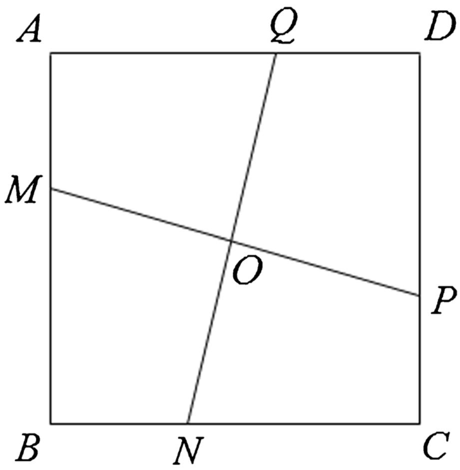
Researcher: How did you introduce to students the idea of geometric transformation in your lesson?
Teacher: The basic principle was to let students experience the process of transformation in solving geometric questions instead of demonstrating the transformation approach myself. However, students should know the basic characteristics of transformation before using them. So I first clarified and introduced the basic ideas and properties of the three transformations (translation, rotation and reflection). Using examples in instructions was necessary. The comparison of similar examples was necessary as well, which was also a kind of reinforcement in students' learning.
Researcher: How did you introduce the use of geometric transformation in constructing auxiliary lines in solving proof problems?
Teacher: Firstly, I led students to explore the impact of translation, rotation and reflection on basic graphs. For example, let students tell the changes after a translation of a triangle: the corresponding sides and angles are unchanged; only the position of the triangle changes. Secondly, I led students to think about the advantages of a transformation in solving proof problems. For instance, students should know that using translation can sometimes make questions easier. Thirdly, for adding auxiliary lines, I guided the students to analyze the existing conditions and observe the characteristics of a figure to reason which transformation can be helpful to the question then to add the corresponding auxiliary lines. In this way, students can gain experiences of adding auxiliary lines from the perspective of geometric transformation.
Researcher: What did you do when students encountered problems in solving the questions?
Teacher: I let them talk to their peers first and then discuss in groups to get different ideas from each other. Finally I summarized different solutions of their discussion and emphasized the solution from a transformation perspective, in which way students can understand the advantages of geometric transformation in solving geometric questions.
Researcher: After the intervention, did you assign exercises for consolidating?
Teacher: Yes. I assigned two similar questions after every lesson.

The following is an intervention classroom episode transcribed from the video-recorded data, which shows how the intervention with IntQ6 was carried out, consistent with the general procedure the teacher described earlier.

IntQ6 In square $ABCD$, M, N, P and Q are the points on sides AB, BC, CD and DA, respectively. $MP = NQ$. Prove $MP \perp NQ$ (Fig. 8).

To solve this proof problem, students need to construct auxiliary lines, which is quite challenging. So the teacher stressed that when no congruent triangles could be easily found, the students need to think if geometric transformation including translation, rotation and folding (reflection) could be used to help. The following is the conversation between the teacher and the students.

Fig. 8 Diagram for IntQ6



Student A: (Connecting *MN* and *QP*), are △*MON* and △*POQ* congruent? Can △*MON* be flipped over to get △*POQ*? No, it is not possible. So reflection does not help.
Student B: It appears that one of the quadrilaterals *AMPD* and *CPMB* can be obtained by rotation of the other. But no, they are not congruent. So it does not help.
Teacher: Apart from reflection and rotation, which cannot really help here, we can consider the third transformation or translation using the given condition *MP*=*NQ* to construct right-angled triangles.
Student C: [Now I know] we can construct auxiliary lines through points *M* and *Q*, so they are parallel to *AD* and *BC,* and then prove that two triangles obtained are congruent.
Teacher: So we used the translation of transformation, translating *AD* to get *ME* and translating *AB* to get *NF* (Fig. 9). In general, we can use translation to construct parallel lines, and obtain equal angles.

According to the teacher, in solving this problem, a critical step was the use of translation to construct two congruent triangles and find the relationships between angles formed with parallel lines. Hence, a good understanding of geometric transformation is very helpful in solving such problems, and a teacher can play an important role in guiding students develop such an understanding, as indicated earlier in the teacher's interview.

The following is another example as revealed by the teacher during the interview, depicting how she used the intervention question to show students how to use geometric transformation approach in solving proof problems (also based on the video-records).

IntQ7: In square *ABCD* shown below, *E* is a point on the side of *CD* and *F* is a point on the side of *AD*. *FB* is the bisector of ∠*ABE*. Prove: *BE* = *AF* + *CE* (Fig. 10).

Teacher: The segment AF and CE are not on the same line. So what should we do?
Student D: Link E and F.

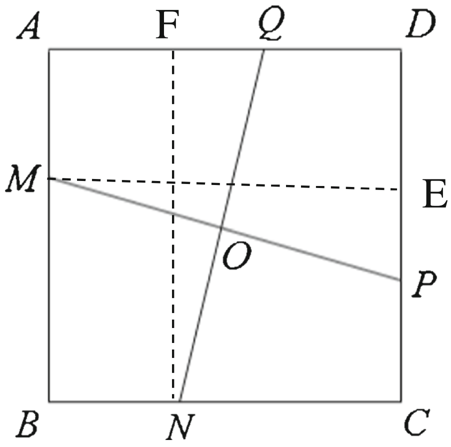


Fig. 9 Auxiliary lines for IntQ6

Teacher: But they are still on two different lines. Here, geometric transformations can be helpful to construct auxiliary lines. Will a reflection make the two segments onto the same line?
Student D: No, here [we] should use a rotation.
Teacher: Yes, when you rotate, be careful about the angle and direction of the rotation.
Student E: Rotate △*BAF* 90 degrees anticlockwise around point B.
Teacher: So is △*BAF* still inside the square?
Student E: No. (after drawing the image of △*BAF*, △*BCG*, see Fig. 11) A goes to C, F goes to G, so △BAF becomes △BCG.

Teacher: Brilliant! The rotation moves all the related sides into one triangle, which is the most important step in solving this question. Adding auxiliary lines (BG and CG in this case) is just an expression of the geometric transformation [So now we just need to prove *BE* = *EG* or ∠*EBG*=∠*BGE*]

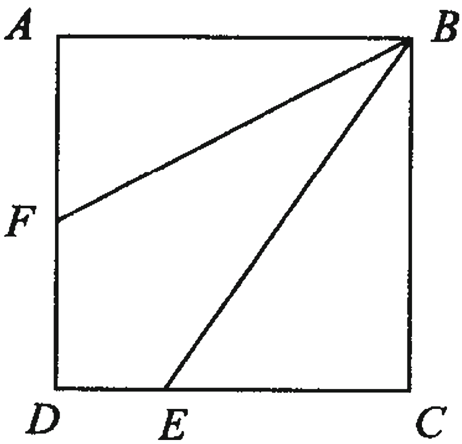
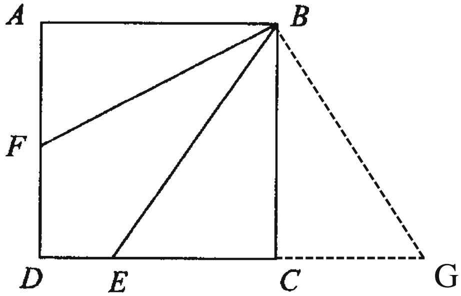


Fig. 10 Diagram for IntQ7

Fig. 11 Diagram showing $\triangle BCG$ is the image of $\triangle BAF$ after rotation



The following excerpts from the interview show how the teacher felt about the change in the experimental class in using transformational approach in the learning of geometry. It is interesting as well as encouraging to note that the teacher offered a rather positive evaluation of the intervention.

> **Researcher**: What did students feel about the use of geometric transformation? Are there any changes in them?
> **Teacher**: Yes. … Firstly, after the intervention, [my recent experience is] 80% of the students in the experimental class first chose the approach of geometric transformation when facing proof problems, while there is only about 30% in the control class. Secondly, about 65% of the students in the experimental class can use the terminology "translation", "reflection" and "rotation" accurately in oral communication, while that was only 8% in the control class. Thirdly, about 70% of the students in the experimental class reported to me that the use of geometric transformation saved the time of constructing auxiliary lines. Fourthly, about 60% of the students in the experimental class felt that the use of geometric transformation is helpful, making solving proof problems easier and faster.

From the teacher's interview, it appears that the intervention made a difference in students' approach to solving geometric problems and had a positive influence on students' use of geometric transformation. It appears further that most students also had a positive view about the use of geometric transformation in solving proof problems.

4 Summary and conclusion

This paper reports an intervention-based study to explore whether a transformation approach in teaching of geometry can improve students' ability in constructing auxiliary lines and hence enhance their learning of solving geometric problems with focus on proof problems and, in particular, high-level cognitive problems. A classroom-based intervention was carried out with a quasi-experimental design in two Chinese secondary classrooms in a two-week duration.

The results of the study based on the data collected from the pre- and post-tests as well as the teacher interview showed a neutral to positive impact of the intervention. On the one hand, there appears no statistically significant difference overall in the impact of using geometry transformation on students' ability in solving general geometric problems, which could be due to the fact that the intervention had a short duration and covered a limited set of geometric topics. On the other hand, encouraging evidence was found to support the use of geometric transformation in solving geometric problems by adding auxiliary lines and hence in enhancing students' learning of geometry. This is particularly evident in students' solutions of challenging geometric questions and in the teacher's observation as reported from the interview data. As described earlier, in solving high-level cognitive geometric questions, the use of transformation helped students realize more clearly why and how auxiliary lines should be added in order to effectively solve the problem (Yang & Pan, 1996; Wang, 2010), and hence enhance their ability in solving these challenging geometric problems.

The study is a first step in our effort to address the difficulty in teaching and learning of geometric proof with a focus on classroom pedagogy (see also Fan, Mailizar, Alafaleq, & Wang, 2016); it was intended to be an exploratory study and hence has some limitations as explained earlier. In future, we think research in two directions is worth undertaking. The first is research of a more confirmatory nature, especially a study with a larger sample size and different groups of students (e.g., in different countries or in different school settings), a wider or different coverage of geometric contents and a longer duration of intervention, though such a study would be also more challenging to implement. The second is research with a focus on high cognition level or challenging geometric proof problems, as one result of this study is that a transformation approach appears to support students' ability in adding auxiliary lines to solve high-level cognitive geometric problems including proof problems and hence enhance their learning of geometry.

Acknowledgements The authors wish to thank Mr. Xianfeng Shi and Ms. Hong Li for their assistance in conducting this study and the anonymous reviewers for their helpful comments. The study was supported in part by a research grant from Beijing Advanced Innovation Centre for Future Education (Project No. BJAICFE2016SR-008).

References

Barbeau, E. J. (1988). Which method is best? *The Mathematics Teacher*, 87–90.

Bonotto, C. (2007). The Erlangen program revisited: A didactic perspective. *For the Learning of Mathematics*, *27*(1), 33–38.

Brändström, A. (2005). *Differentiated tasks in mathematics textbooks. An analysis of the levels of difficulty.* Luleå: Luleå University of Technology. http://epubl.luth.se/1402-1757/2005/18/LTU-LIC-0518-SE.pdf. Accessed 16 October 2015

China Ministry of Education. (2001). *Mathematics curriculum standards for compulsory education (experimental version)*. Beijing: Beijing Normal University Press.

China Ministry of Education. (2012). *Mathematics curriculum standards for compulsory education* (2011 ed.). Beijing: Beijing Normal University Press.

Chou, S., Gao, X., & Zhang, J. (1994). *Machine proofs in geometry: Automated production of readable proofs for geometry theorems (Vol. 6)*. Singapore: World Scientific.

Coxford, F. A. (1973). A transformation approach to geometry. In K. B. Henderson (Ed.), *Geometry in the mathematics curriculum: Thirty-sixth yearbook*. Washington, DC: National Council of Teachers of Mathematics.

Common Core State Standards Initiative. (2009). *Common Core State Standards*. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf. Accesses 15 Sept. 2016

Department for Education. (2013). *Mathematics programmes of study: Key Stage 3 (National curriculum in England)*. https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/239058/SECONDARY_national_curriculum_-_Mathematics.pdf. Accessed 14 December 2014.

Department for Education. (2014). *Mathematics programmes of study: Key Stage 4 (National curriculum in England)*. https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/331882/KS4_maths_PoS_FINAL_170714.pdf. Accessed 16 October 2015

Fan, L., Mailizar, M., Alafaleq, M., & Wang, Y. (2016). *How proof is presented in selected secondary maths textbooks in China, Indonesia and Saudi Arabia? Paper presented in the 13th Internatonal congress on mathematics education*. Germany: Hamburg.

Gao, F. (2010). Exploring rotational problems in equilateral triangles. *Mathematics in Primary and Secondary Schools (Junior Secondary School Edition), 11*, 29–30.

Golzy, J. (2008). *A cultural study of classroom discourse and its impact on students' initiation of geometry proofs* (Doctoral dissertation). Available from ProQuest Dissertations and Theses. (UMI No. 3320455)

Harel, G. (1999). Students' understanding of proofs: A historical analysis and implications for the teaching of geometry and linear algebra. *Linear Algebra and Its Application*, *302–303*, 601–613.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 805–842). Charlotte: Information Age Publishing.

Herbst, P., & Brach, C. (2006). Proving and doing proofs in high school geometry classes: What is it that is going on for students? *Cognition and Instruction*, *24*(1), 73–122.

Hiebert, J., Gallimore, R., Garnier, H., Givvin, K. B., Hollingsworth, H., Jacobs, J., … Stigler, J. (2003). *Teaching Mathematics in Seven Countries: Results from the TIMSS 1999 Video Study*. Washington, DC: U.S. Department of Education, National Center for Education Statistics. http://nces.ed.gov/pubs2003/2003013.pdf. Accessed 16 October 2015.

Hodds, M., Alcock, L., & Inglis, M. (2015). Self-explanation training improves proof comprehension. *Journal for Research in Mathematics Education*, *45*(1), 62–101.

Hollebrands, K. F. (2003). High school students' understandings of geometric transformations in the context of a technological environment. *The Journal of Mathematical Behavior*, *22*(1), 55–72.

Hoyles, C., & Jones, K. (1998). Proof in dynamic geometry contexts. In C. Mammana & V. Villani (Eds.), *Perspectives on the teaching of geometry for the 21st century* (pp. 121–128). London: Springer.

Inglis, M., & Alcock, L. (2012). Expert and novice approaches to reading mathematical proofs. *Journal for Research in Mathematics Education*, *43*(4), 358–390.

Jones, K. (2000). Critical issues in the design of the school geometry curriculum. In B. Barton (Ed.), *Readings in mathematics education* (pp. 75–90). Auckland: University of Auckland. http://eprints.soton.ac.uk/41335/. Accessed 16 October 2015

Ma, F. (Ed.). (2014a). *Mathematics (grade 7–9)*. Beijing: Beijing Normal University Press.

Ma, F. (Ed.). (2014b). *Mathematics (grade 9, volume 1)*. Beijing: Beijing Normal University Press.

Mariotti, M. A. (2006). Proof and proving in mathematics education. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 173–204). Rotterdam, the Netherlands: Sense Publishers.

Marrades, R. (2000). Proofs produced by secondary school students learning geometry in a dynamic computer envrironment. *Educational Studies in Mathematics*, *44*(1–2), 87–125.

McCrone, S. M. S., & Martin, T. S. (2004). Assessing high school students' understanding of geometric proof. *Canadian Journal of Math, Science & Technology Education*, *4*(2), 223–242.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston: Author.

Nissen, P. (2000). A geometry solution from multiple perspectives. *The Mathematics Teacher*, *93*(4), 324.

Ng, B. Y., & Tan, S. L. (1984). From Euclidean geometry to transformation geometry. *The Mathematica Medley*, *12*(2), 57–66.

Senk, S. L. (1985). How well do students write geometry proofs? *The Mathematics Teacher*, *78*(6), 448–456.

Stein, M. K., & Smith, M. S. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics teaching in the middle school*, *3*(4), 268–275.

Stylianides, A. J., & Stylianides, G. J. (2013). Seeking research-grounded solutions to problems of practice: Classroom-based interventions in mathematics education. *ZDM–International Journal on Mathematics Education*, *45*(3), 333–341.

Usiskin, Z. P. (1972). The effects of teaching Euclidean geometry via transformations on student achievement and attitudes in tenth-grade geometry. *Journal for Research in Mathematics Education*, *3*(4), 249–259.

Usiskin, Z. P., & Coxford, A. F. (1972). A transformation approach to tenth-grade geometry. *The Mathematics Teacher*, *65*(1), 21–30.

Usiskin, Z. (2014). Transformations in US commercial high school geometric textbooks since 1960: A brief report. In K. Jones, C. Bokhove, G. Howson, & L. Fan (Eds.), *Proceedings of the international conference on mathematics textbook Research and Development (ICMT-2014)* (pp. 471–476). Southampton: Southampton Education School, University of Southampton.

Wang, X. (2010). *A research on exercise teaching of geometric figures and transformation in junior mathematics* (Master's thesis). Soochow University, China. http://max.book118.com/html/2014/0406/7333382.shtm. Accessed 31 July 2015

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*(1), 101–119.

Willson, W. W. (1977). *The mathematics curriculum: Geometry*. London: Blackie.

Yang, Q., & Pan, S. (1996). The application of three types of transformations in adding auxiliary lines. *Journal of Suzhou Institute of Education, Issue No.*, *1*, 63–65.

Yao, X. (2010). The geometric transformation and auxiliary lines in squares. *Essential Readings for Junior Secondary School Students*, Issue No.1, 34–36.

You, A. (2009). The difficulties in geometry and some solutions. *Secondary School Mathematics, 12*, 8–9.